Generalized Canonical Transform Method

M. E. Gorbunov

A. M. Obukhov Institute of Atmospheric Physics RAS, Moscow, Russia

G. Kirchengast

Wegener Center for Climate and Global Change, University of Graz, Graz, Austria

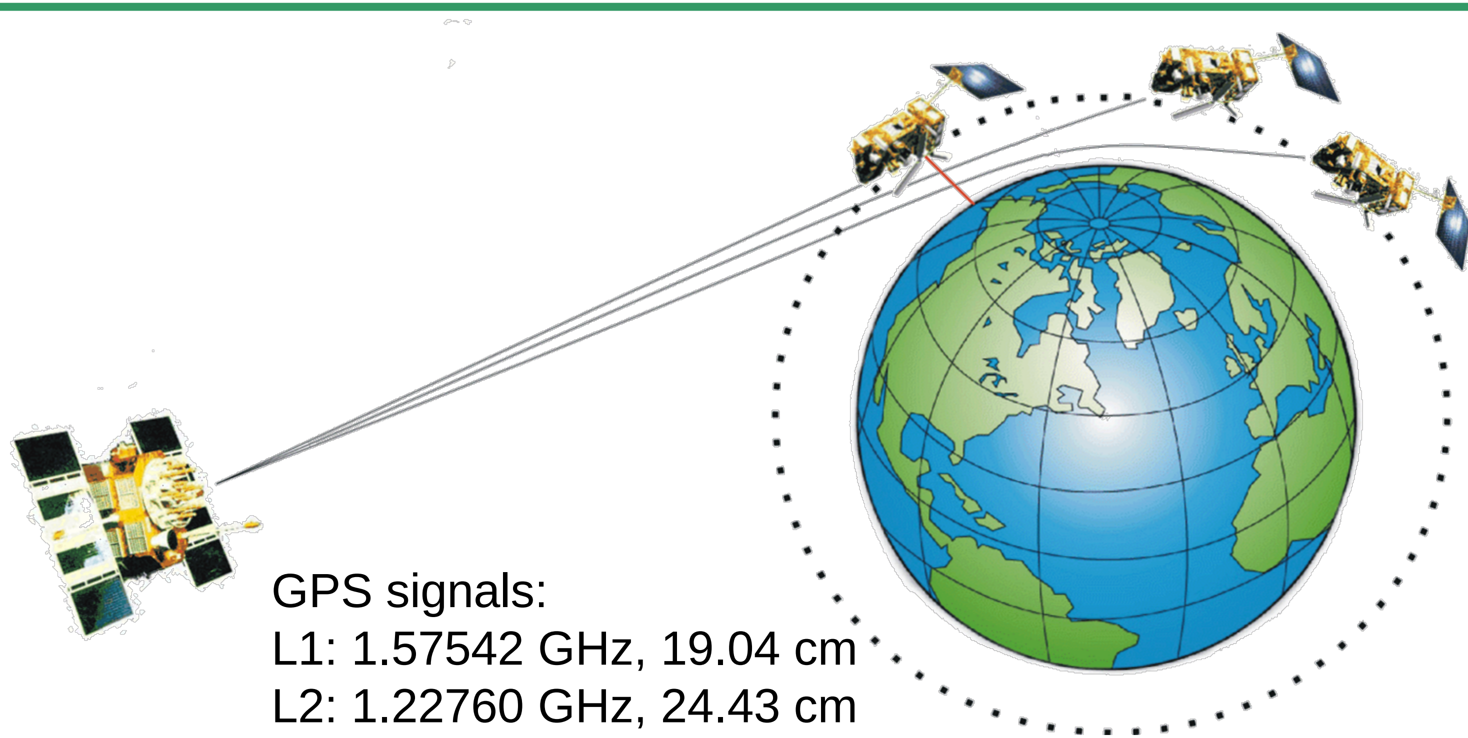
K. B. Lauritsen

Danish Meteorological Institute, Copenhagen, Denmark

Contents

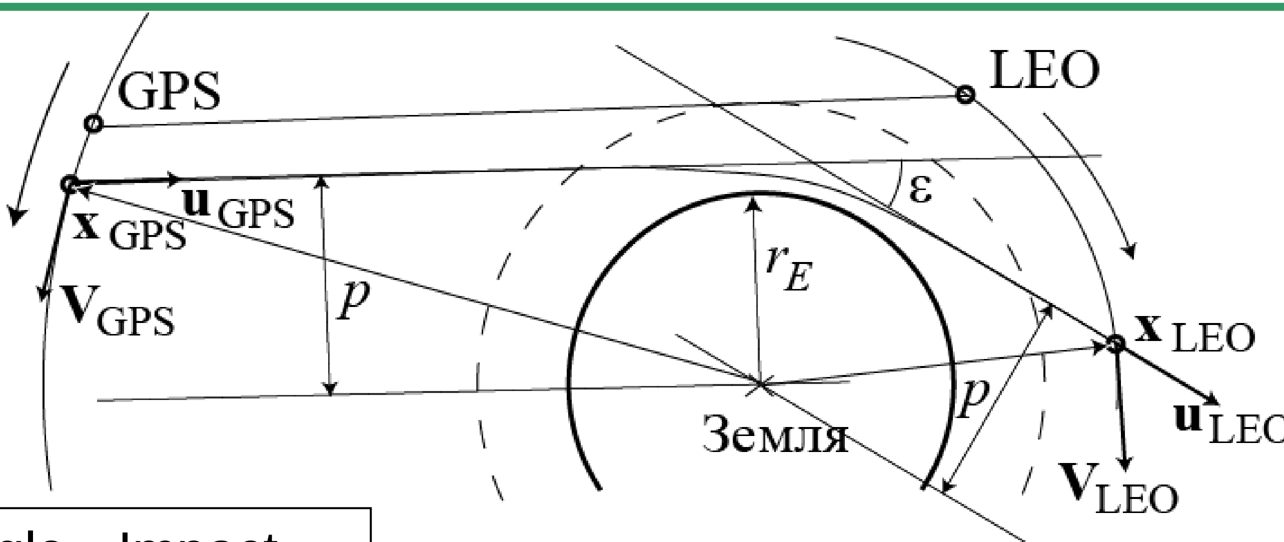
- Radio occultation sounding of the Earth's atmosphere
- Linear representations of wave fields and the reconstruction of the ray manifold
- What is a Canonical Transform
- Back Propagation as Canonical Transform
- Standard **C**anonical **T**ransform complemented with an **A**ffine transform (CTA)
- Reflected ray retrieval as an example of CTA
- Statistical analysis of COSMIC data with CTA

Radio occultations



1. Measurements of refraction of radio signals of GNSS (GPS, GLONASS, Galileo) in the Earth's atmosphere on limb paths.
2. Retrieval of vertical profiles of temperature, pressure and humidity.

Radio occultations



Bending angle ε Impact parameter *p*

$$\varepsilon(p) \xleftrightarrow{\text{Linear transform}} \ln n(x)$$

Refractivity *n*

Distance from the Earth's center *r*

Refractive radius *x*

$$\varepsilon(p) = -2p\int_{r_0}^{\infty} \frac{d\ln n(r)}{dr}\frac{dr}{\sqrt{n(r)^2 r^2 - p^2}}$$

$$n(x) = \exp\left(\frac{1}{\pi}\int_{p}^{\infty}\frac{\varepsilon(p)dp}{\sqrt{p^2 - x^2}}\right)$$

$$x = n(r)r; \quad r(x) = \frac{x}{n(x)}$$

Basic variables

Measured quantities

$A_{1,2}(t)$ Amplitude [V/V]

$\Psi_{1,2}(t)$ Atmospheric phase excess [m]

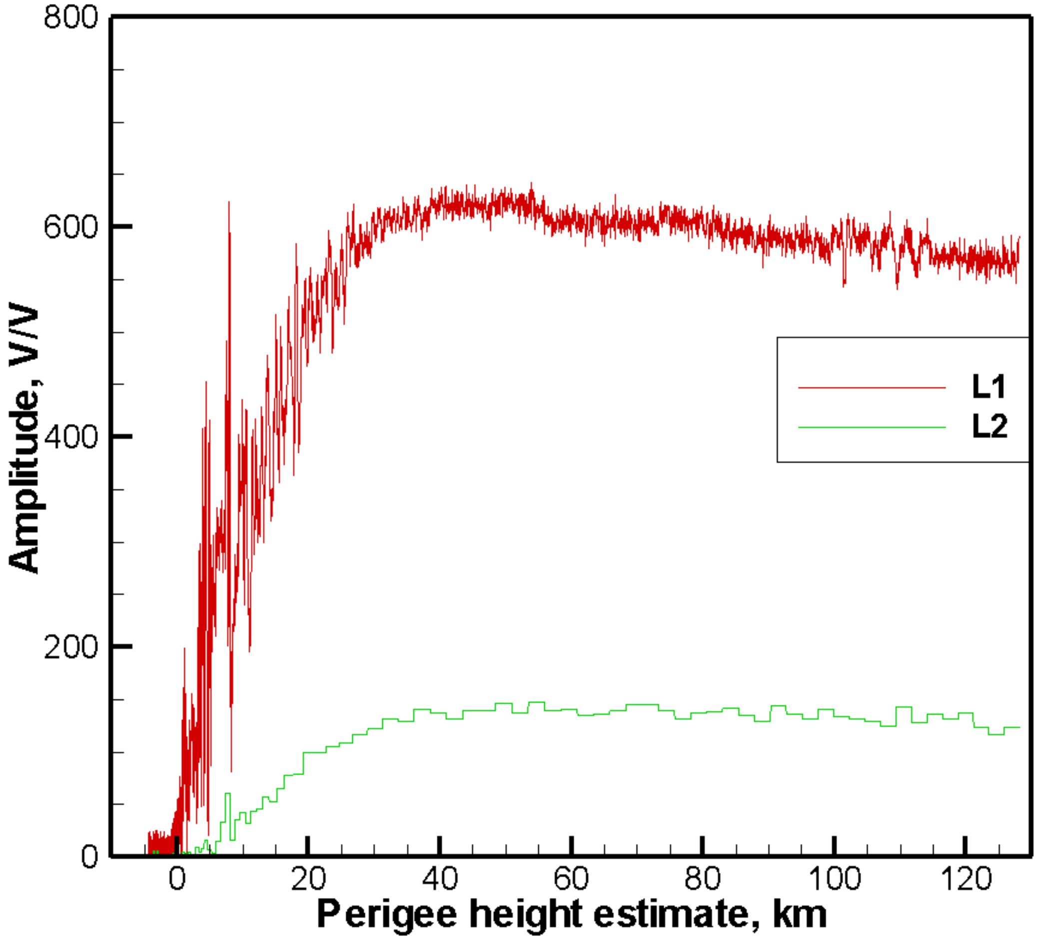
Difference between the full optical path and GPS–LEO distance

$\left.\begin{matrix}\mathbf{r}_{GPS}(t) \\ \mathbf{r}_{LEO}(t)\end{matrix}\right\}$ Satellite orbit data

$\Psi_M(t)$ Phase excess model

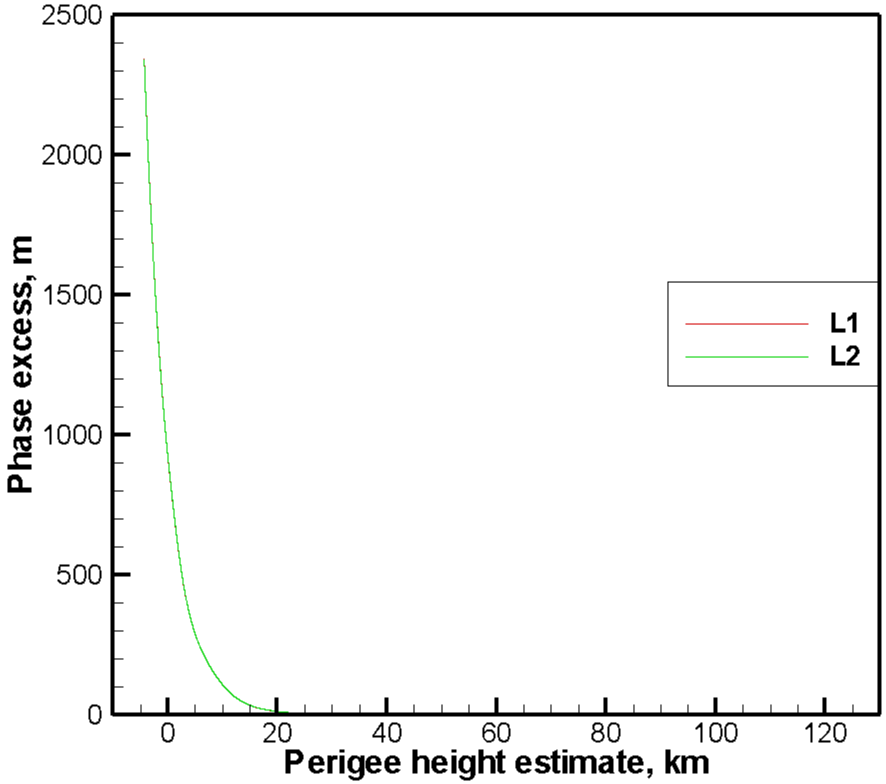
It is evaluated for a climatic atmospheric model and describes the Doppler frequency shift with an accuracy of 10–15 Hz

Example of amplitude

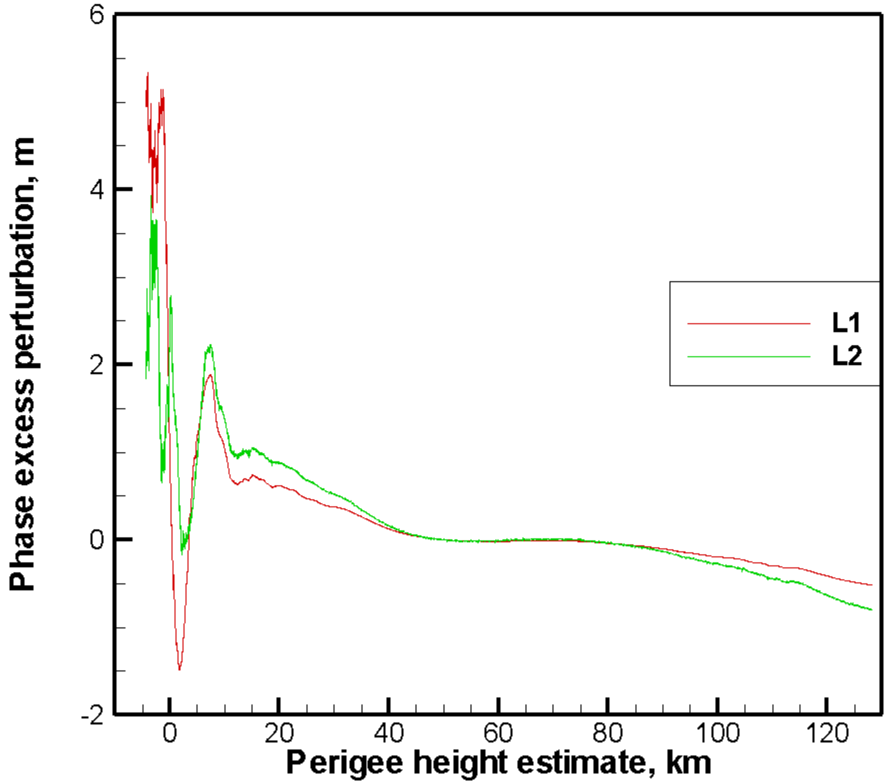


Example of phase excess

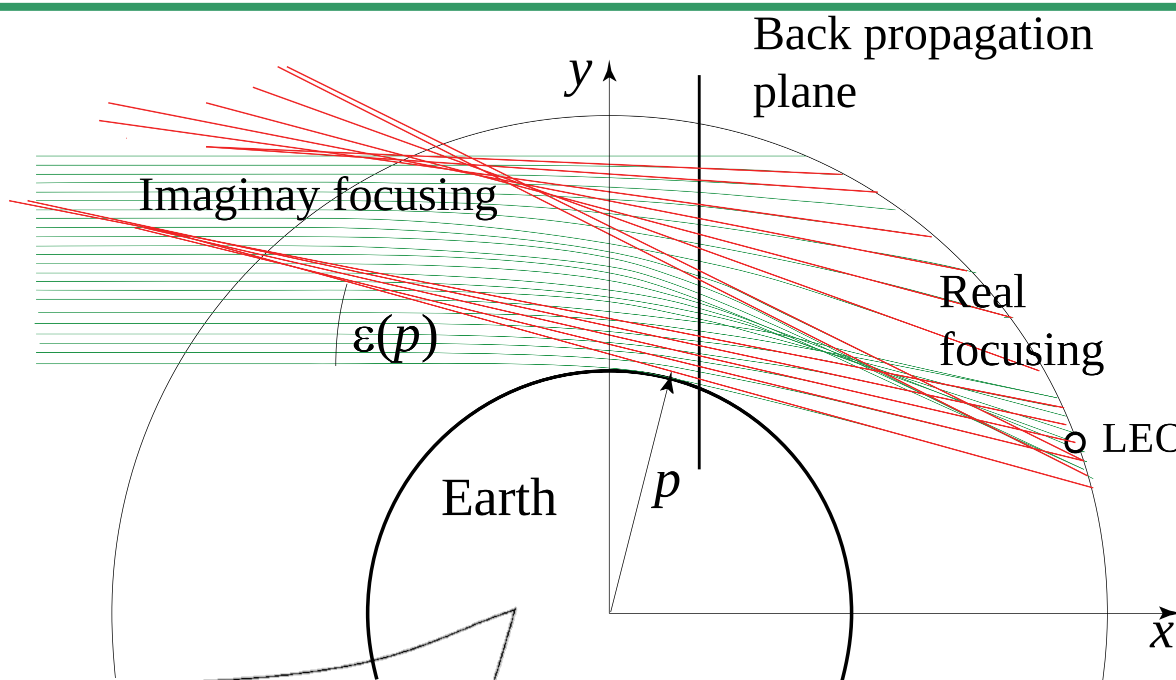
$$\Psi_{1,2}(t)$$

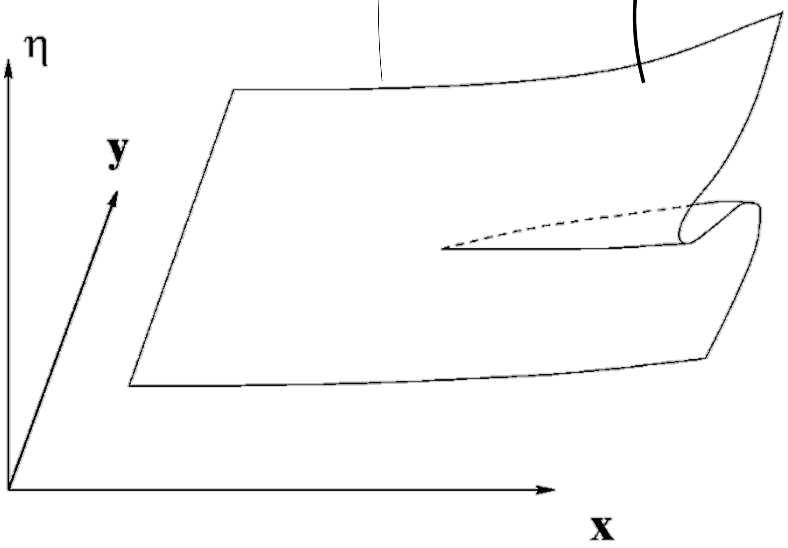


$$\Psi_{1,2}(t) - \Psi_M(t)$$



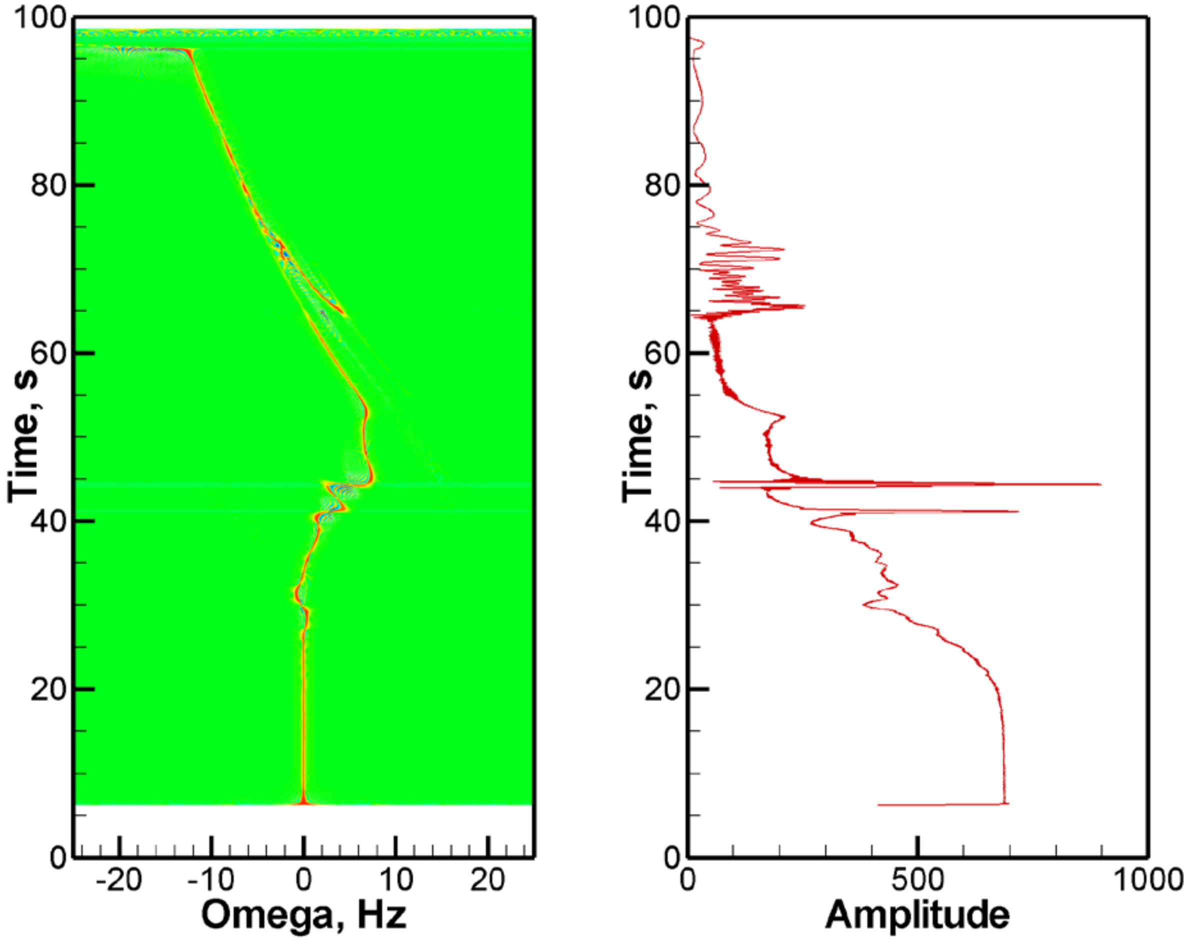
Multipath propagation





For the retrieval of ray manifold structure in multipath zones, we used approaches introduced by V.A.Fock, V.P.Masolv, Yu.V.Egorov, and L. Hörmander and developed the Canonical Transform method.

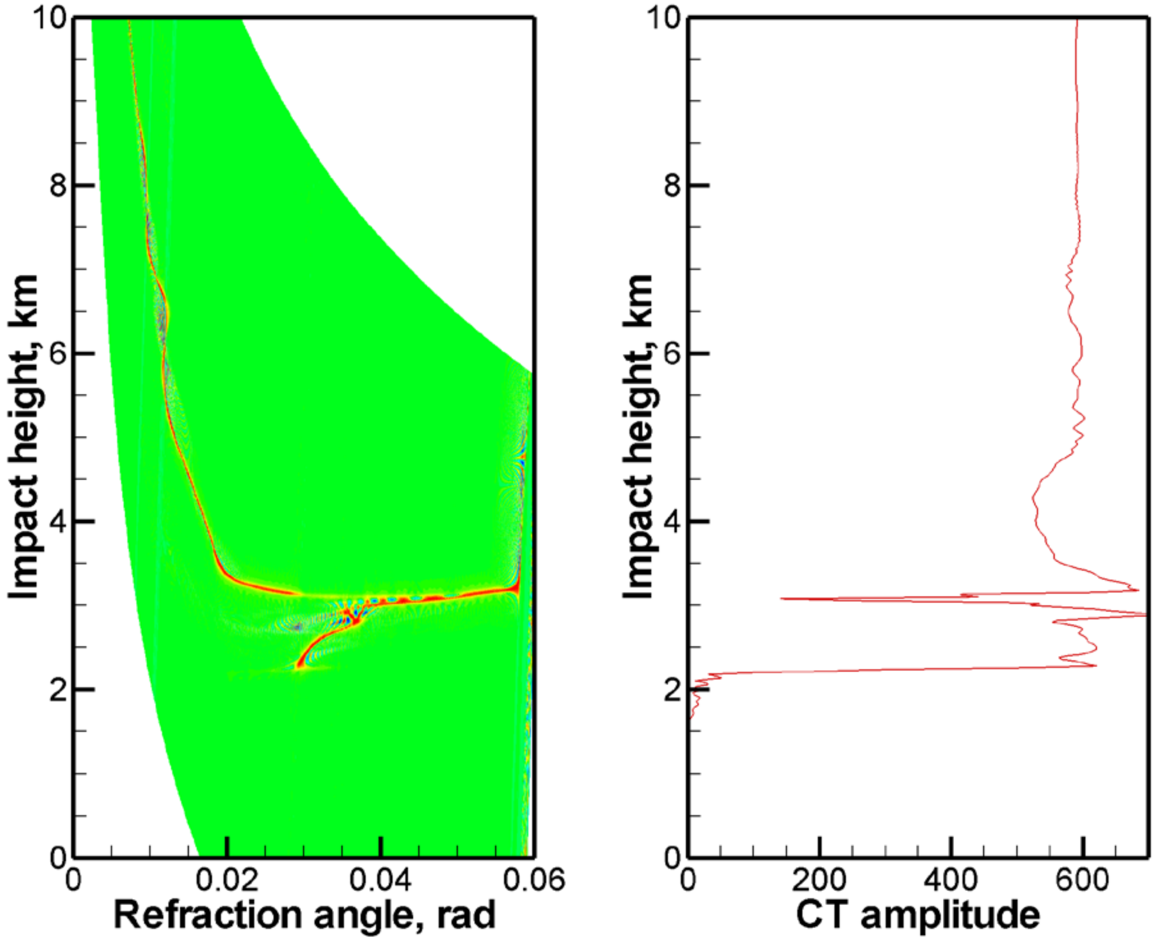
Projections of ray manifold



$$A(t) = \frac{\Delta E}{\Delta t}$$

Amplitude is the energy density wrt time

Projections of ray manifold



$$A'(p) = \frac{\Delta E}{\Delta p}$$

CT amplitude is the energy density wrt impact height

Canonical Transforms

Wave field in t- and p-representation:

$$u(t) = \sum A_i(t) \exp\left[ik \int \sigma_i(t)\, dt \right],$$

$$\hat{u}(p) = \sum A_i'(p) \exp\left[ik \int \varepsilon_i(p)\, dp \right]$$

$k = \dfrac{2\pi}{\lambda}$ - wave number

$\sigma = -\dfrac{\omega}{k} = \dfrac{d\Psi}{dt}$ - derivative of the eikonal

Linear transform between the representations

$$\hat{u}(p) = \int K(p,t)\, u(t)\, dt$$

corresponds to the canonical transform

$$(t,\sigma) \to (p,\varepsilon)$$

conserving the volume element of the ray (phase) space:

$$dt \wedge d\sigma = dp \wedge d\varepsilon$$

$$\varepsilon\, dp - \sigma\, dt = dS(p,t) \qquad K(p,t) = \sqrt{\mu(p,t) \frac{\partial^2 S(p,t)}{\partial p\, \partial t}} \exp\left(ikS(p,t)\right)$$

Canonical Transforms

Canonical transform is a linear reversible transformation of the wave field.
The transform is expressed in terms of Fourier Integral Operator (FIO), which is a linear operator with the oscillating kernel.

$$\hat{u}(z) = \int \sqrt{\mu(y,x)\frac{\partial^2 S(z,s)}{\partial z\,\partial s}} \exp\left(ikS(z,s)\right) u(s)\,ds$$

$$u(s) = \sum A_i(s) \exp\left[ik\int \xi_i(s)\,ds\right],$$

$$\hat{u}(z) = \sum A_i'(z) \exp\left[ik\int \eta_i(z)\,dz\right]$$

$$ds \wedge d\xi = dz \wedge d\eta$$

$$\eta\,dz - \xi\,ds = dS(z,s)$$

Egorov's theorem

Canonical transform is a linear reversible transformation of the wave field. The transform is expressed in terms of Fourier Integral Operator (FIO), which is a linear operator with the oscillating kernel.
Canonical transform is a representation of the wave field. This means that the dynamic equation for the field is also transformed.

The canonical transform:

$$(y, \hat{\eta}) \to \left(z, \hat{\xi}\right),$$

where momenta are associated with differential operators.
The original dynamic equation:

$$-\frac{1}{ik}\frac{\partial u}{\partial x} = H(x, y, \hat{\eta})u,$$

where H is the Hamilton function and x is the propagation coordinate.
The transformed equation:

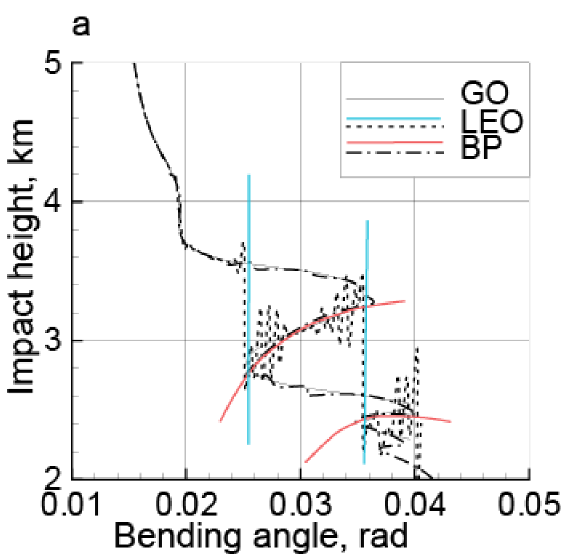
$$-\frac{1}{ik}\frac{\partial \widehat{\Phi} u}{\partial x} = H\left(x, y\left(z, \hat{\xi}\right), \eta\left(z, \hat{\xi}\right)\right)\widehat{\Phi} u,$$

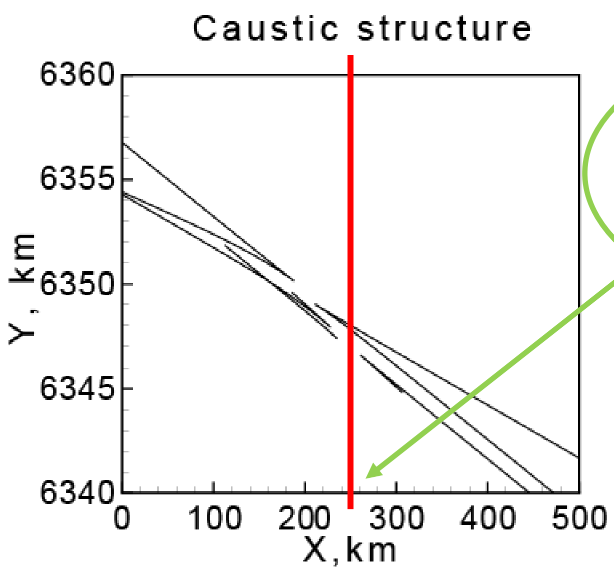
where $\widehat{\Phi}$ is the FIO associated with the canonical transform.

Canonical Transforms

Back Propagation is also a canonical transform.

$$\hat{u}(\mathbf{r}_B) = \sqrt{\frac{ik}{2\pi}} \int_S u_0(\mathbf{r}) \cos\varphi(\mathbf{r}, \mathbf{r}_B) \frac{\exp\left(-ik\left|\mathbf{r}-\mathbf{r}_B\right|\right)}{\left|\mathbf{r}-\mathbf{r}_B\right|^{1/2}} dS_{\mathbf{r}},$$

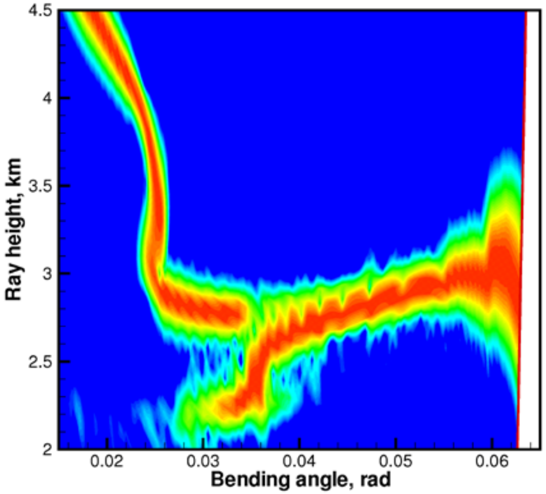


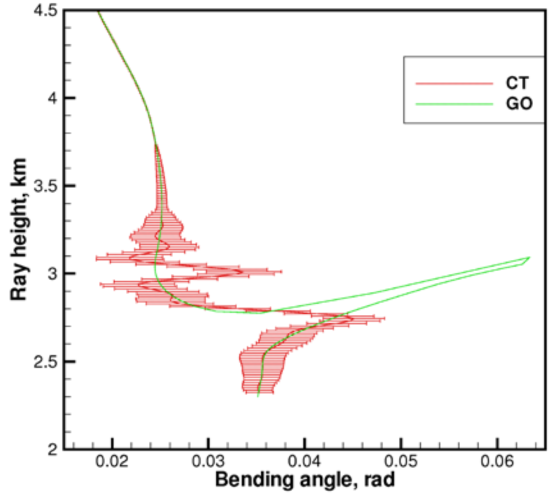


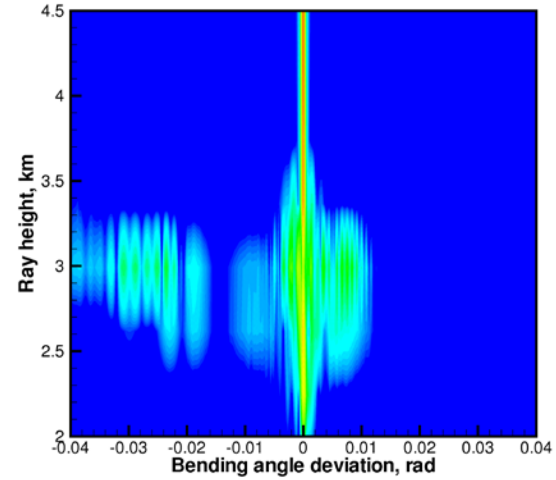
The transform parameter:
x_B - the BP plane position

This type of transform has non-optimal coordinate lines in the phase space.
The standard CT (FSI, PM, CT2) use the impact height as the new coordinate.

Example: BA

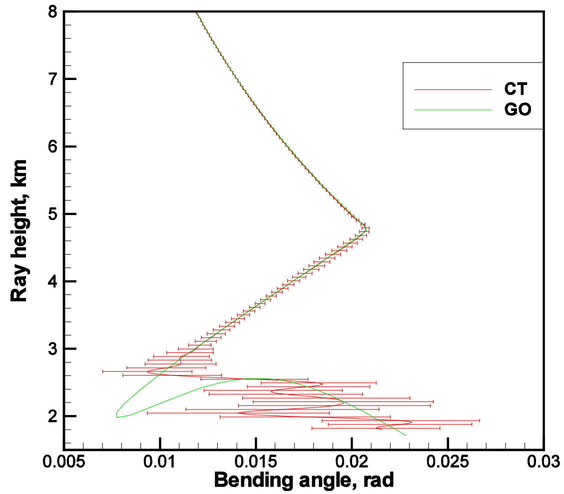


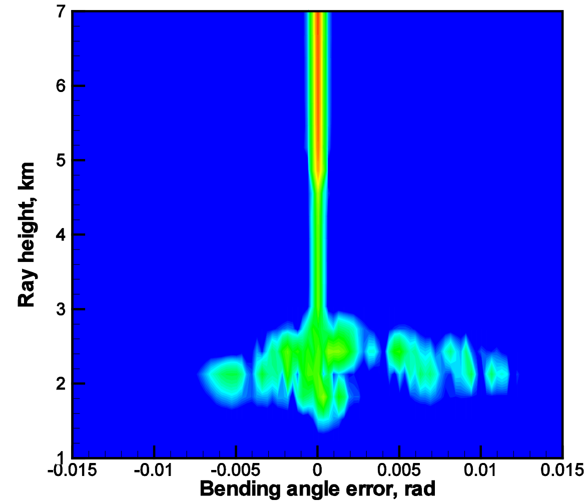




Simulations based on fixed ECMWF field: 5 Feb 1997, UTC 00:00

Example: Sokolovskiy front





Sokolovskiy front: MWR 133, 2200 (2005):

$$N(z,\theta) = N_0 \exp\left(-\frac{z}{H}\right)\left[1 + \mu \sin\left(\frac{\pi z}{2\varepsilon}\right) - R_e \frac{\theta}{d}\right],$$

$N_0 = 340$ N−units, $H = 7.5$ km, $\mu = 0.15$, $\varepsilon = 0.15$, $d = 15.0$ km, $z = r - R_e$

Error Estimate of Bending Angles in the Presence of Strong Horizontal Gradients 23

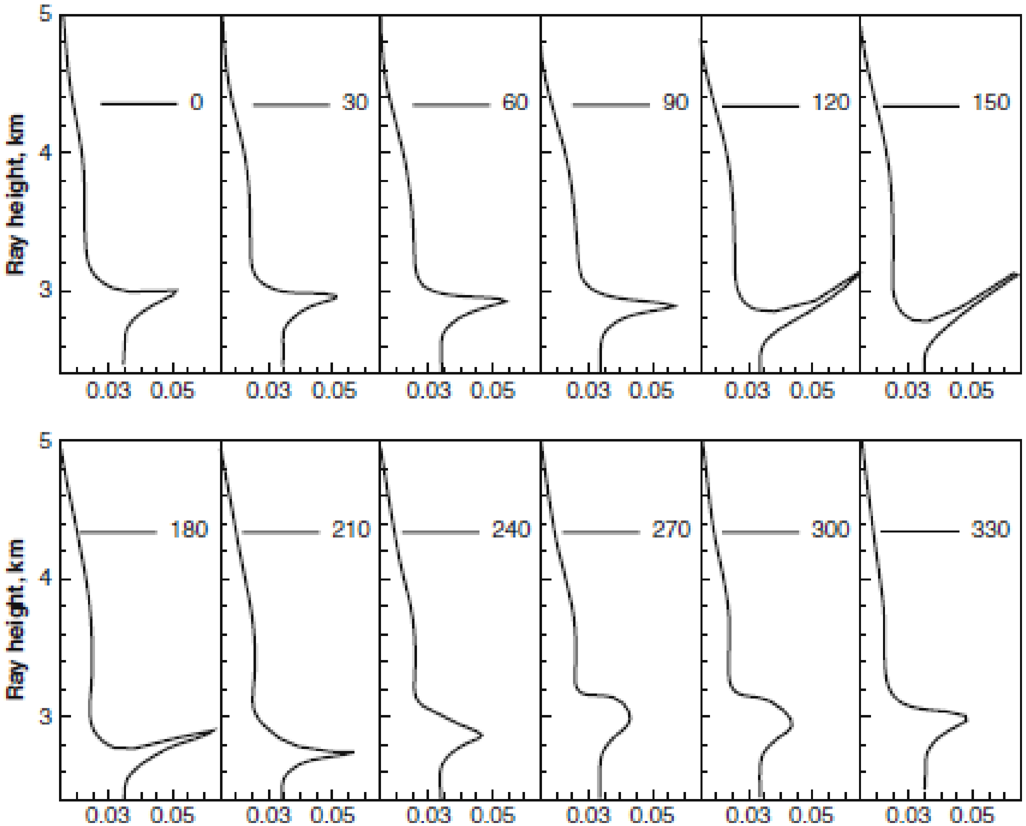


Fig. 3 True simulated bending angle profiles for sounding of the same location 12.4°N, 170.9°W from different azimuths (from 0° to 330°, with a step of 30°). Azimuths are characterized by the angle between the occultation plane and local north direction. The modeling was based on ECMWF fields February 5, 1997, UTC 00:00

Dynamic equation for impact parameter:

$$\frac{dp}{ds} = \frac{\partial n}{\partial \theta}$$

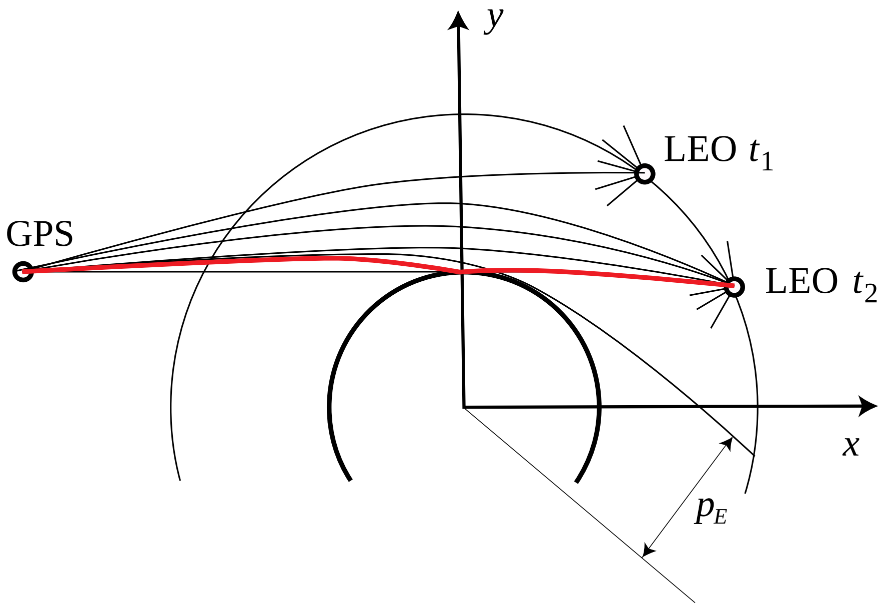
From [M. E. Gorbunov and K. B. Lauritsen, Error Estimate of Bending Angles in the Presence of Strong Horizontal Gradients, in: “New Horizons in Occultation Research. Studies in Atmos-phere and Climate”, editors: A. Steiner, B. Pirscher, U. Foelsche, and G. Kirchengast. – Springer Berlin Heidelberg, 2009, 360 p. – p. 17–26, doi: 10.1007/978-3-642-00321-9_2].

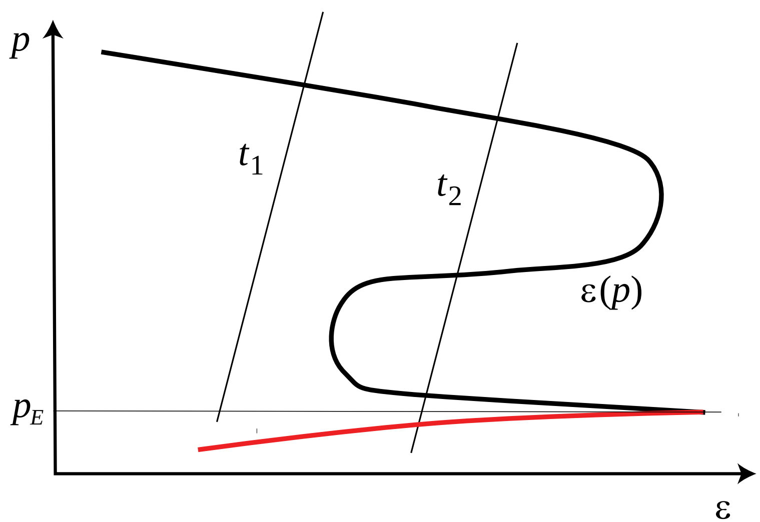
Reflected rays

$$\varepsilon_R(p) = -2p \int_{p_E}^{\infty} \frac{d \ln n}{dx} \frac{dx}{\sqrt{x^2 - p^2}} - 2 \arccos\left(\frac{p}{p_E}\right),$$

$$p_E = r_E n(r_E)$$ r_E – Earth's radius

From the refraction angle profile of reflected rays, it may be possible to estimate the surface refractivity.

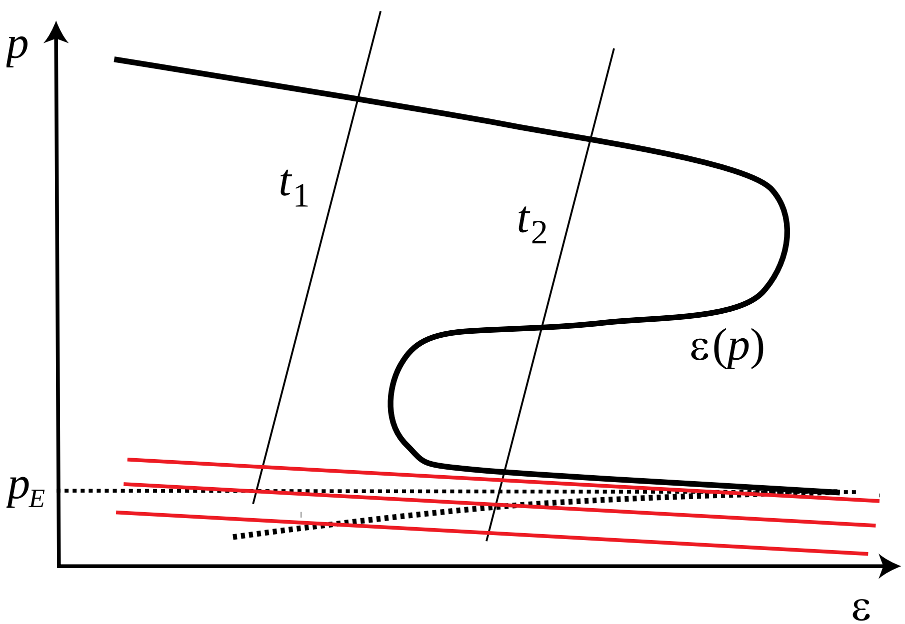




Reflected rays: modified impact height

<u>The CT2 method uses the linearized form of Phase Matching transform</u>

$p_0(t)$ - smooth model of impact parameter
$\sigma_0(t)$ - smooth model of eikonal derivative



$\Upsilon(t) = \int \left(\frac{\partial p_0}{\partial t}\right)^{-1} dt$ - new coordinate

$$\delta\Upsilon = \delta\varepsilon(p)$$

$\sigma = \frac{d\Psi}{dt}$ - old momentum

$\eta = \frac{d\Psi}{d\Upsilon} = \frac{\partial p_0}{\partial \sigma}\sigma$ - new momentum

$f(t) = p_0(t) - \frac{\partial p_0}{\partial t}\sigma_0(t)$ - ancillary function

The original transform:

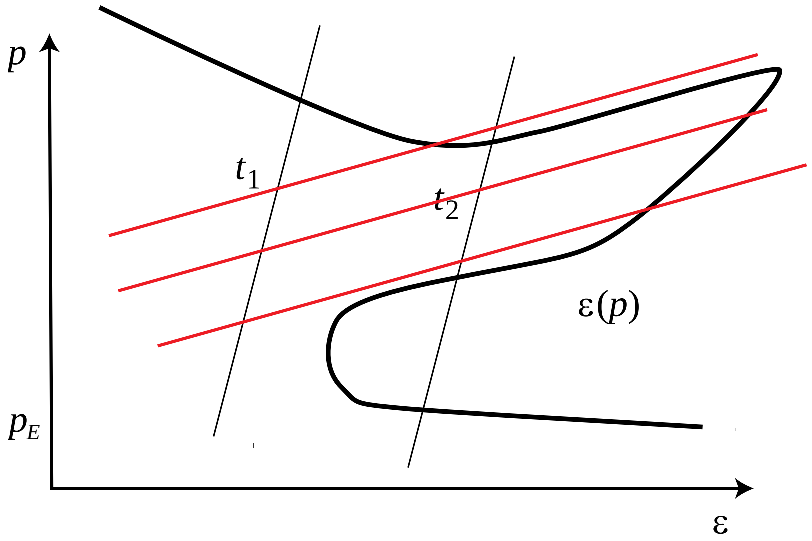
$$p = f(\Upsilon) + \eta$$
$$\xi = -\Upsilon$$

The modified coordinate:

$$p' = p + \beta\Upsilon$$

Horizontal gradients: modified impact height

CTA = a composition of standard CT with an affine transform



Additional affine transform:

$$p' = p - \beta(\xi - \xi_0)$$
$$\xi' = \xi$$

Generating function:

$$dS_\beta(p', \xi) = \xi dp' + p d\xi =$$
$$= \xi dp' + (p' + \beta(\xi - \xi_0))d\xi$$

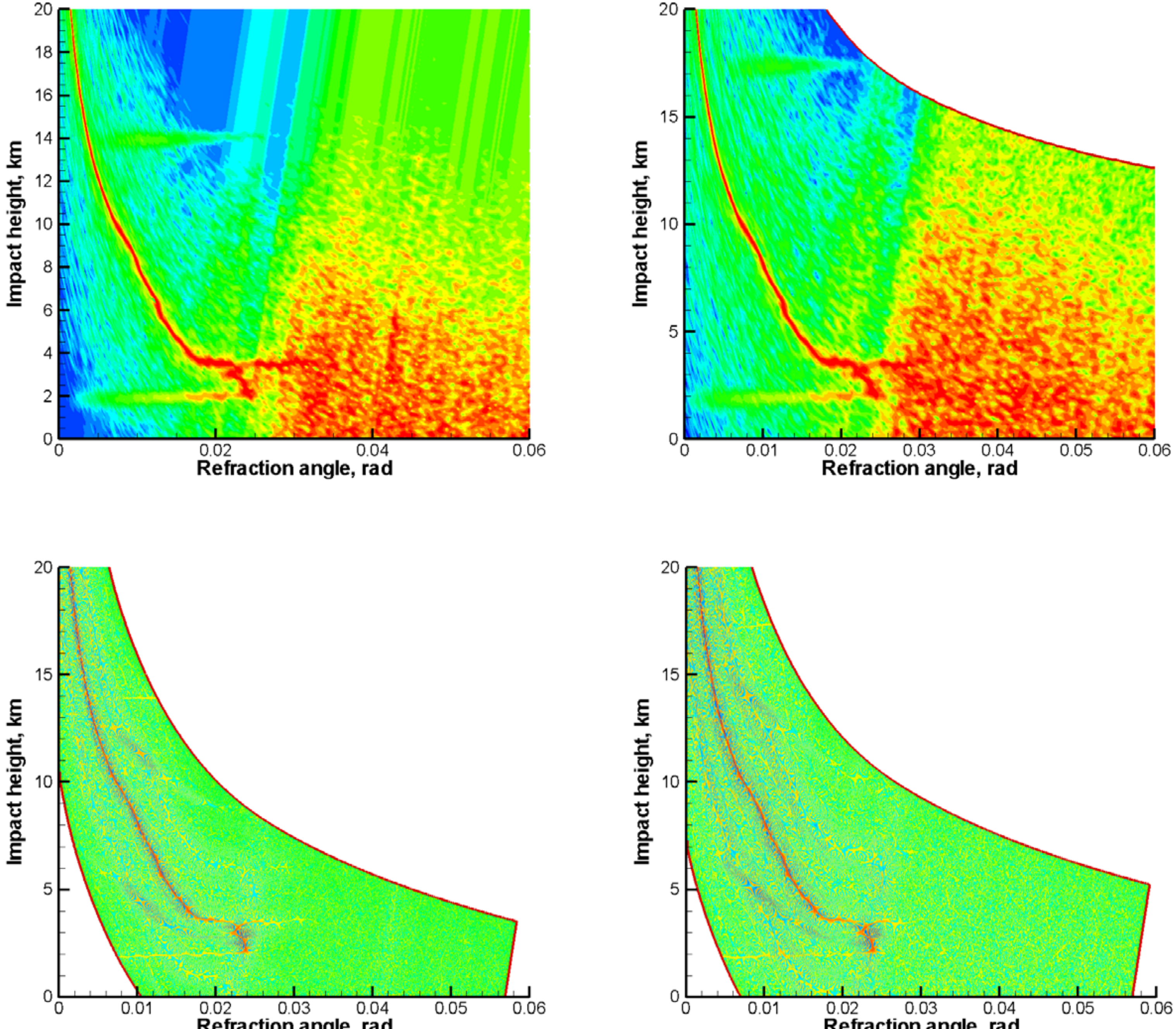
$$S_\beta(p', \xi) = \xi p' + \beta \frac{(\xi - \xi_0)^2}{2}$$

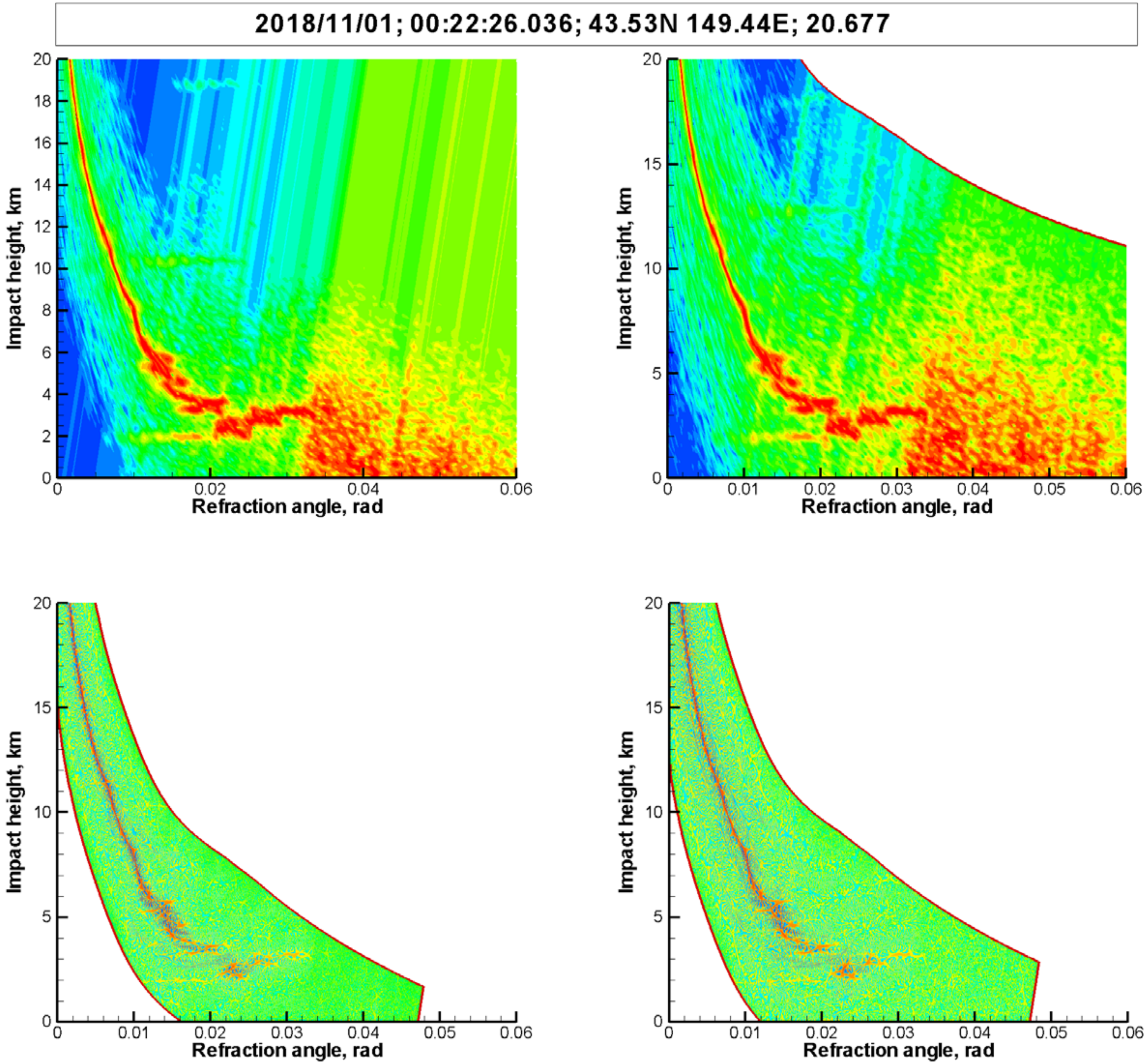
New operator:

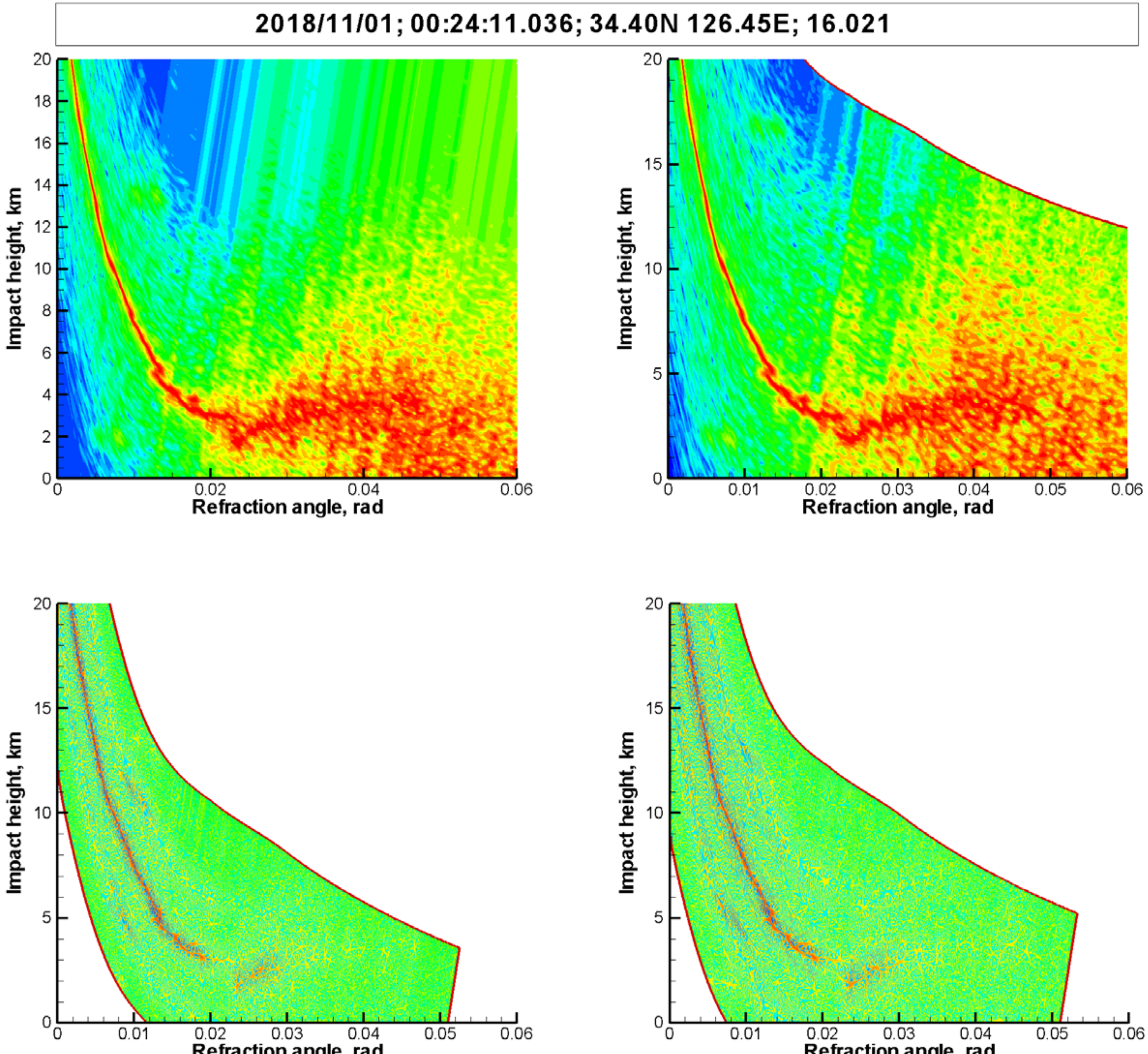
$$\hat{\Phi}_2^{(\beta)}[u](p') = \hat{\Phi}^{(\beta)}\{\hat{\Phi}_2[u]\}(p')$$

$$\hat{\Phi}_2^{(\beta)}[u](p') = \int \sqrt{\frac{\partial^2 S_\beta}{\partial p' \partial \xi}} \exp(ikS_\beta)\tilde{u}(\xi)d\xi = \int \exp(ikS_\beta)\tilde{u}(\xi)d\xi$$

2018/11/01; 00:10:59.035; 82.15N 101.64E; 13.521

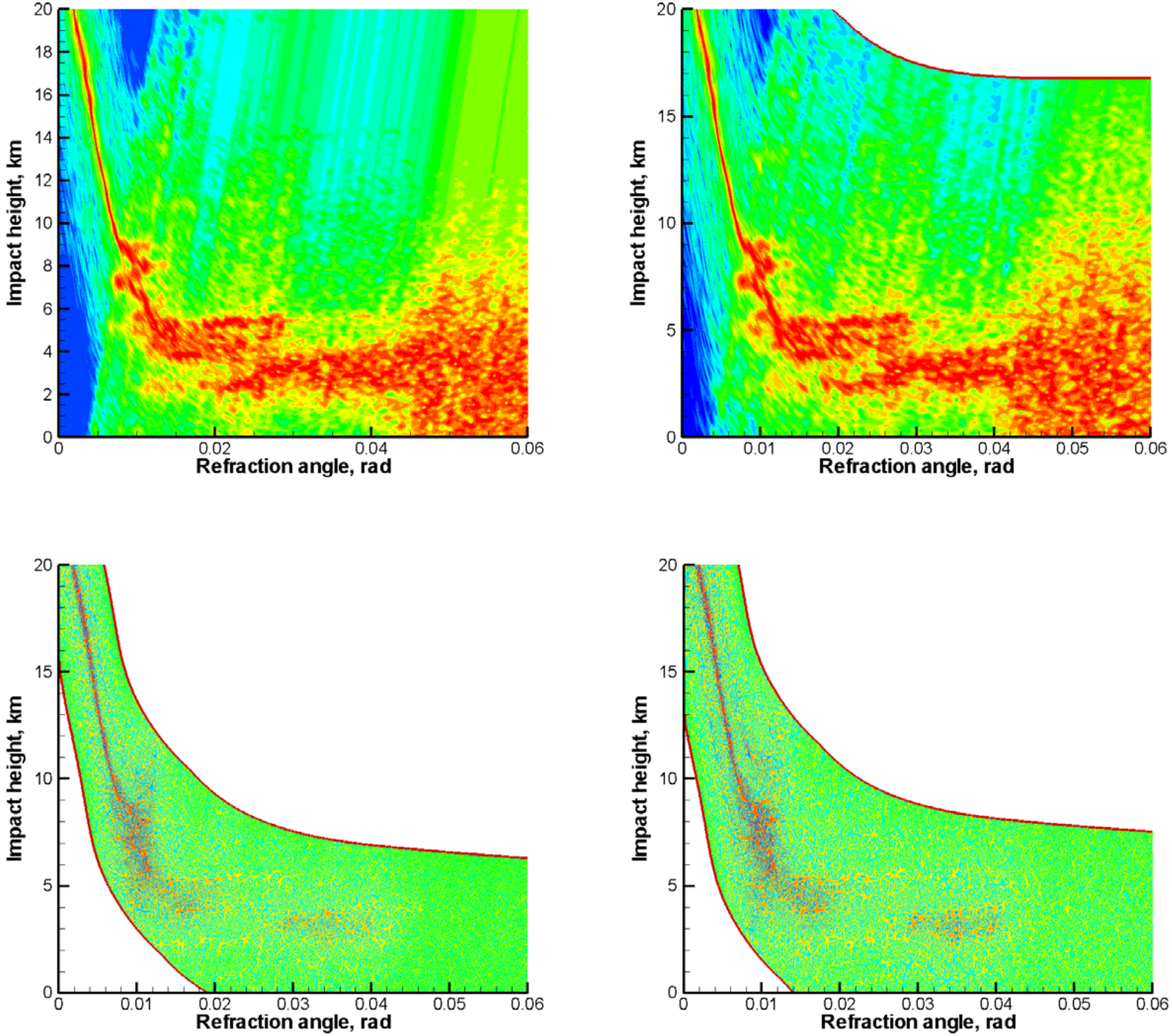


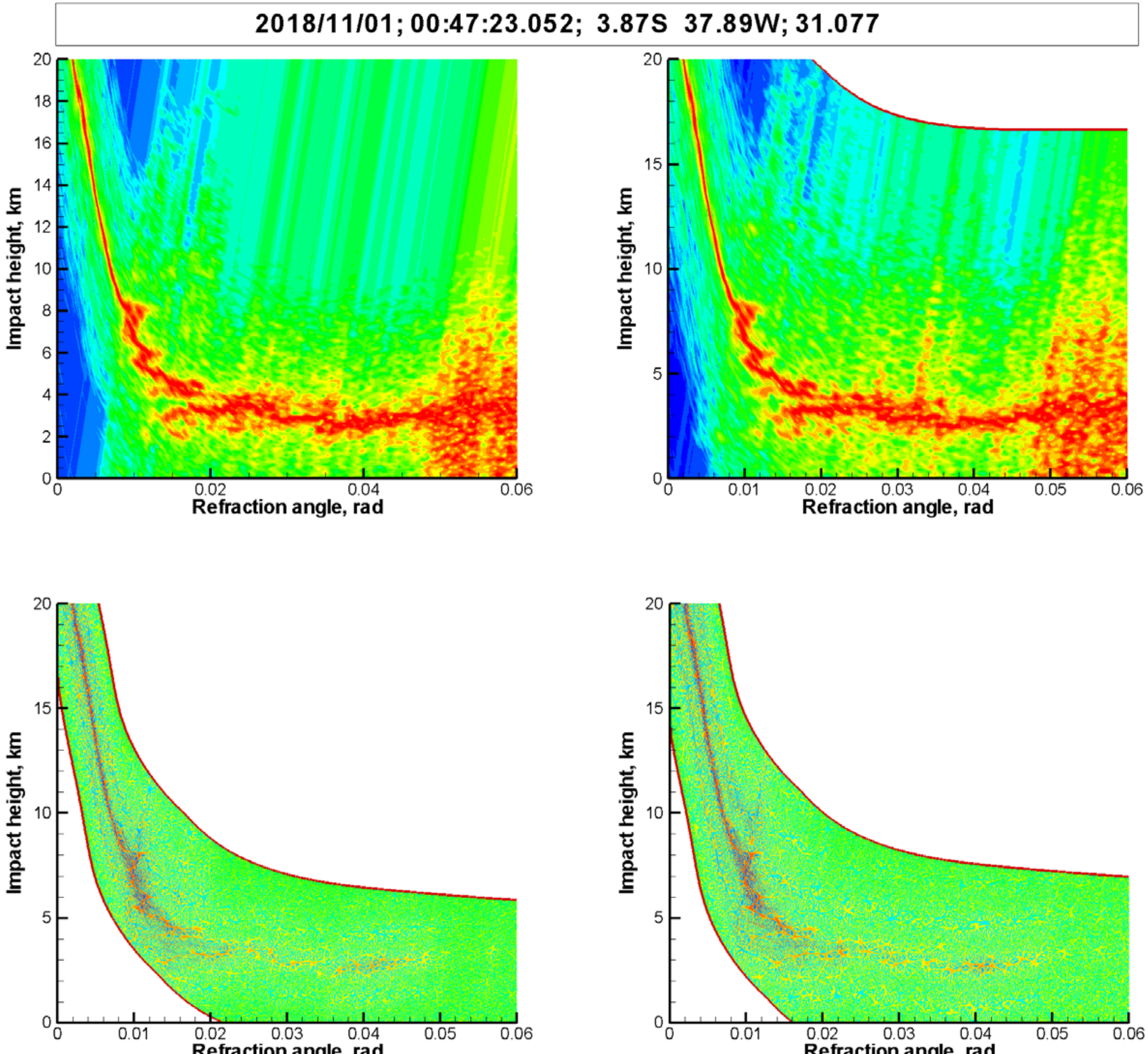

2018/11/01; 00:22:26.036; 43.53N 149.44E; 20.677
Impact height, km
Refraction angle, rad
Impact height, km
Refraction angle, rad
Impact height, km
Refraction angle, rad
Impact height, km
Refraction angle, rad



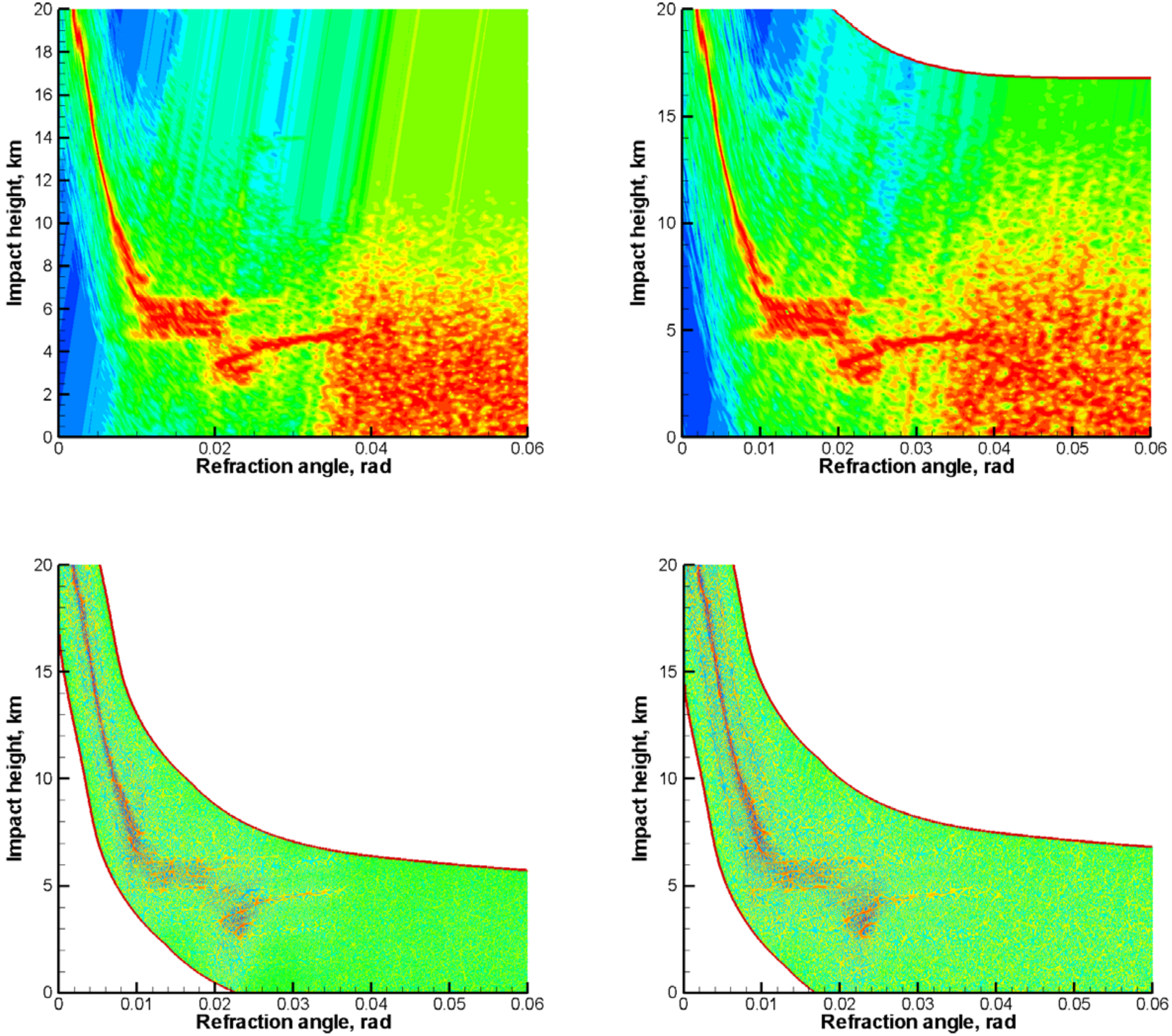
2018/11/01; 00:24:11.036; 34.40N 126.45E; 16.021
Impact height, km
Refraction angle, rad
Impact height, km
Refraction angle, rad
Impact height, km
Refraction angle, rad
Impact height, km
Refraction angle, rad

2018/11/01; 00:47:02.057; 2.36S 33.36W; 16.228

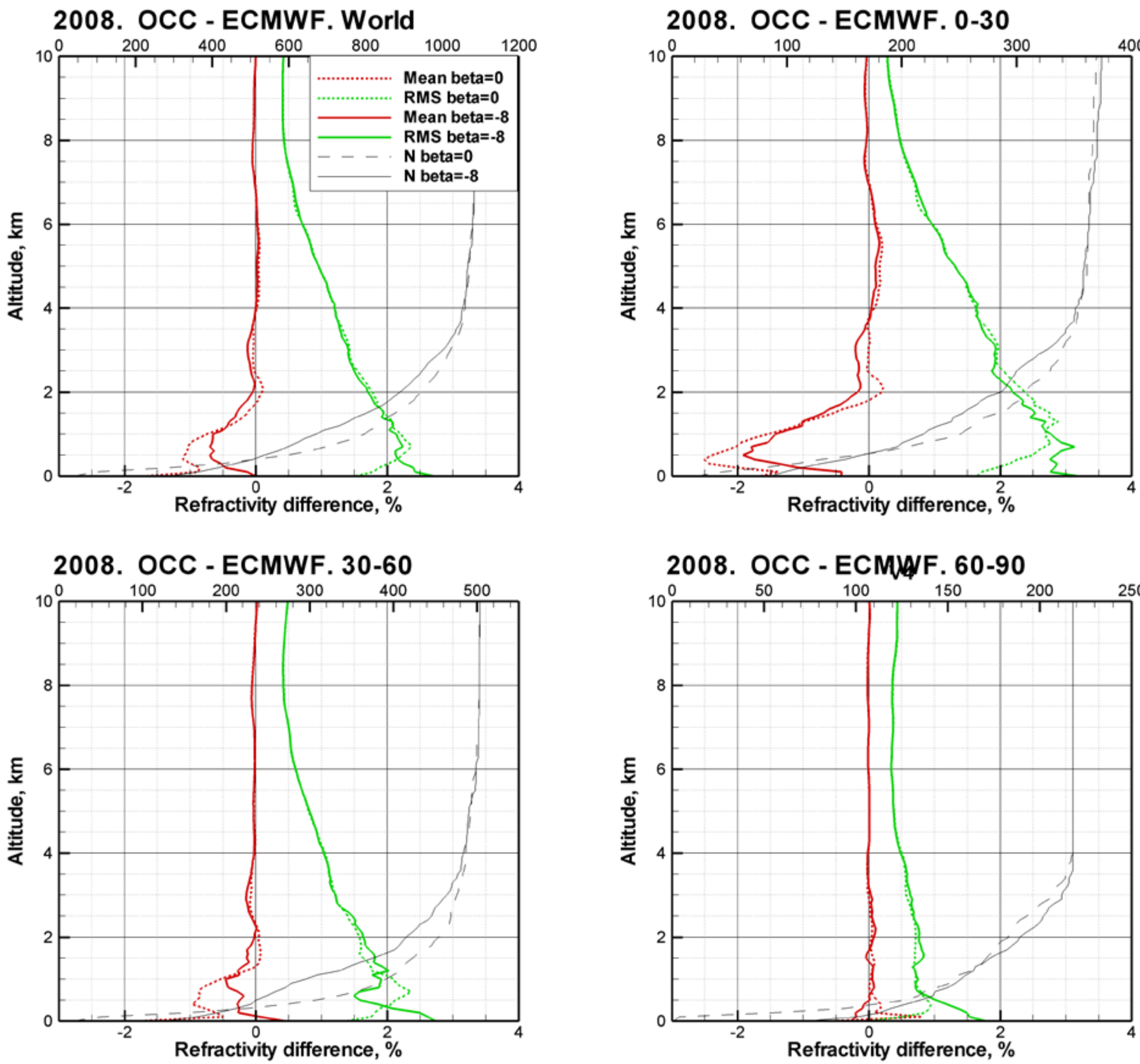



2018/11/01; 00:47:23.052; 3.87S 37.89W; 31.077
Impact height, km
Refraction angle, rad
0
2
4
6
8
10
12
14
16
18
20
5
15
0.01
0.02
0.03
0.04
0.05
0.06

2018/11/01; 01:31:35.052; 14.97S 125.51E; 25.368



Statistical analysis



Conclusions

1. Canonical transforms is a very general class of linear representations of signals.
2. Canonical transforms depend on the choice of coordinates in the phase space associated with the signal.
3. In RO data processing, CT, FSI, PM, and CT2 operators were based on the impact height representation.
4. To better account for the horizontal gradient, a more general choice of coordinate is possible, referred to as CTA: a composition of the standard CT with an affine transform.
5. Statistical tests of CTA indicate an improvement in RO retrievals: decrease of systematic and random errors.
6. The improvement is small, as expected: the time of big solutions has passed.

~~*Mischief managed*~~

Thanks for attention!