Kirkwood Distribution Function and its Application for the Analysis of Radio Occultation Observations

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Introduction

The Wigner Distribution Function (WDF) was introduced very early during the development of quantum mechanics. It appeared in works by Wigner, Dirac, Weyl, and Heisenberg [1-5], who derived it from the wave energy density operator and re-formulated quantum mechanics in terms of the classical phase space. Later, it was also used in signal processing and was as well applied to optics, acoustics, and the analysis of Radio Occultation (RO) observations [6-12]. WDF unfolds a one-dimensional signal, or a function of time, in the time-frequency plane and allows decomposing a signal into multiple quasi-monochromatic tones, i.e., signals with slowly changing amplitude and frequency. This property made it useful for the analysis of RO signals in multi-path zones. Less known is the Kirkwood Distribution Function (KDF), which was also introduced in quantum mechanics in the same time period as WDF [13] .

WDF and KDF have some common properties. Both asymptotically tend to a delta-like distribution concentrated on the ray manifold. They, however, also exhibit some differences. While WDF numerical evaluation requires significant computation costs, KDF evaluation is exceptionally easy and only requires a single (large) Fast Fourier Transform. However, the use of KDF is not that straightforward: unlike WDF, the amplitude of which directly visualizes the ray manifold shape, KDF utilizes the phase relations only: its stationary point lies at the ray manifold and, in proportion to the distance from the ray manifold, its oscillation frequency increases. In this work, we study the utility of the KDF for analyzing RO observations. We give examples of analyzing real RO events (see right part of poster), with the evaluation of both WDF and KDF, and discuss the possible advantages of the latter.

Basic Relations

Wigner Distribution Function (WDF):

$$\rho^W(x,\xi) = \frac{k}{2\pi}\int \psi\left(x-\frac{s}{2}\right)\overline{\psi}\left(x+\frac{s}{2}\right)\exp(iks\xi)\,ds$$

$$= \int \overline{\tilde{\psi}}\left(\xi-\frac{\sigma}{2}\right)\tilde{\psi}\left(\xi+\frac{\sigma}{2}\right)\exp(ikx\sigma)\,d\sigma.$$

x - observational coordinate
ξ - corresponding frequency
k - wavenumber
$\psi(x)$ - observed wave field

Kirkwood Distribution Function (KDF):

$$\rho^{x\xi}(x,\xi) = \frac{k}{2\pi}\int \psi(x)\overline{\psi}(x+s)\exp(iks\xi)\,ds =$$

$$= \sqrt{\frac{-ik}{2\pi}}\psi(x)\exp(-ikx\xi)\sqrt{\frac{ik}{2\pi}}\int \overline{\psi}(\tilde{x})\exp(ik\tilde{x}\xi)\,d\tilde{x}.$$

WDF is linked to KDF by a convolution:

$$\rho^W(x,\xi) = \rho^{x\xi}(x,\xi) * T^W_{x\xi}(x,\xi),$$

$$T^W_{x\xi}(x,\xi) = \frac{k}{\pi}\exp(2ik\,\xi\,x).$$

KDF Properties

1. KDF is a complex, sign-alternating function.
2. Positive projections:
$$\int \rho^{x\xi}(x,\xi)\,dx = \left|\tilde{\psi}_t(\xi)\right|^2,$$
$$\int \rho^{x\xi}(x,\xi)\,d\xi = \left|\psi_t(x)\right|^2.$$
3. Positive full energy:
$$\int \rho^{x\xi}(x,\xi)\,dx\,d\xi = E = \int \left|\psi(x)\right|^2 dx = \int \left|\tilde{\psi}(\xi)\right|^2 d\xi.$$
4. Global univalent phase:
$$\psi(x) = a(x)\exp(ikS(x)),$$
$$\tilde{\psi}(\xi) = \tilde{a}(\xi)\exp(ik\tilde{S}(\xi))$$
$$\rho^{x\xi}(x,\xi) = a^{x\xi}(x,\xi)\exp\left(ikS^{x\xi}(x,\xi)\right),$$
$$S^{x\xi}(x,\xi) = S(x) - \tilde{S}(\xi) - x\xi.$$

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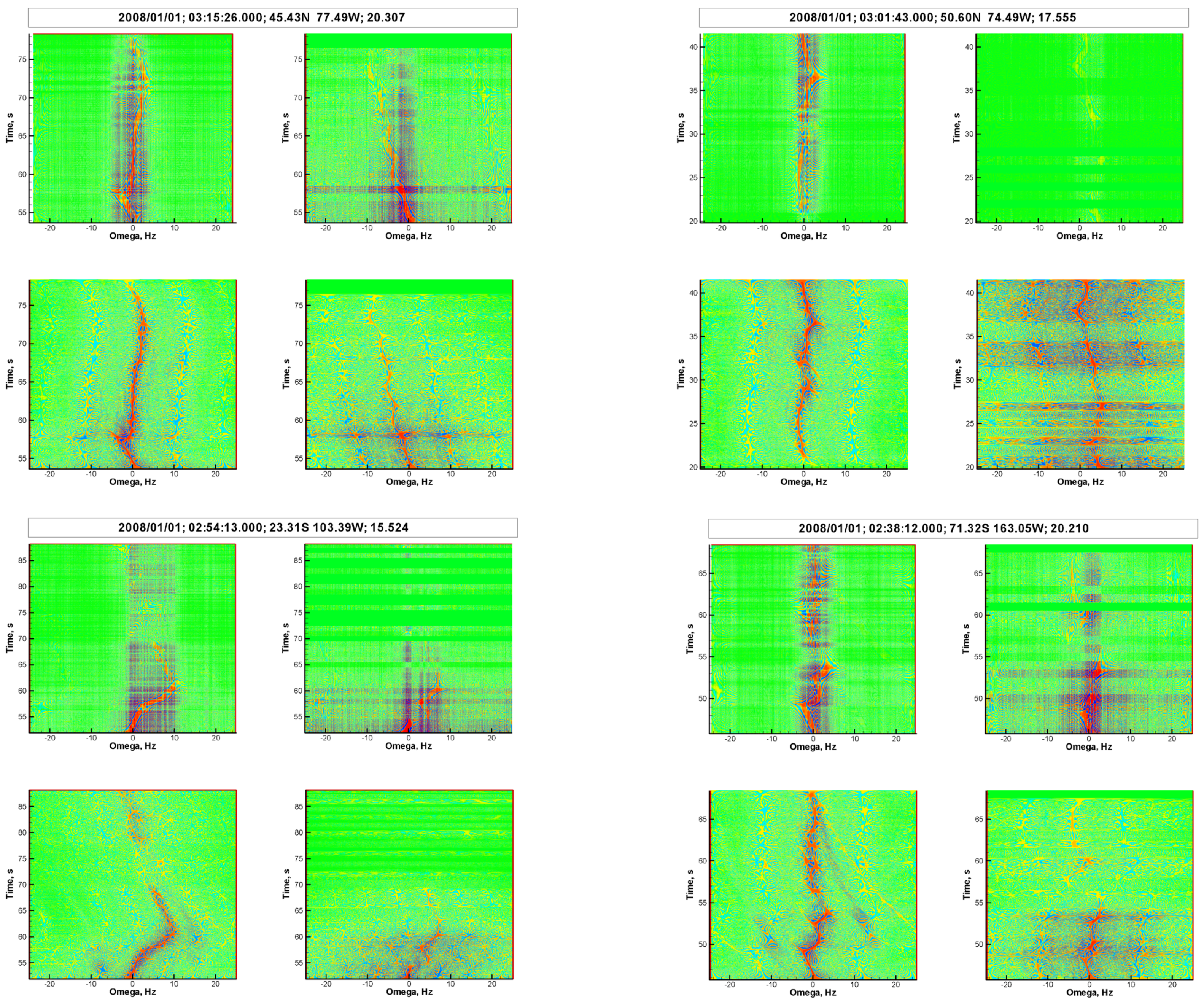
Examples of Analysis Results

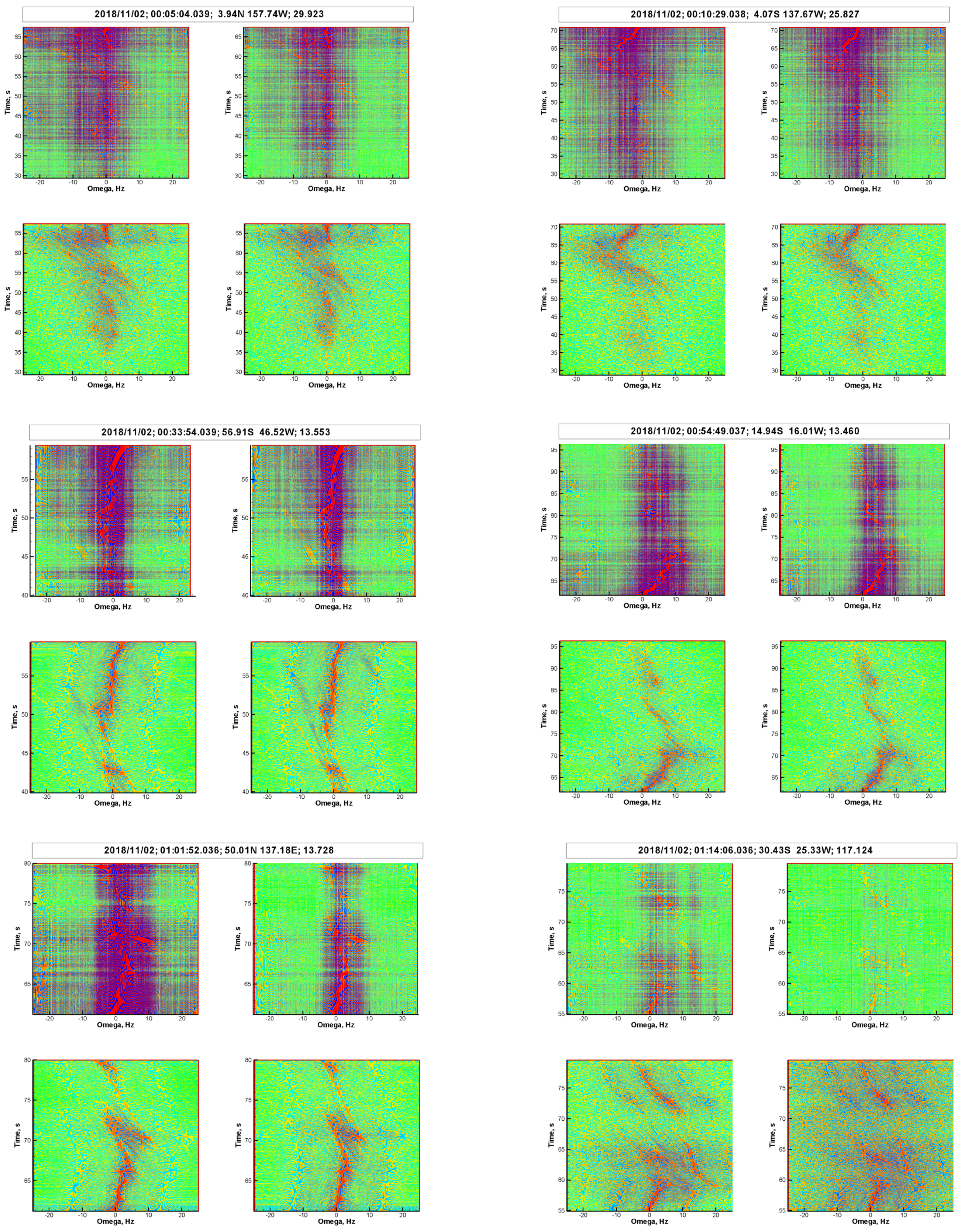
Below we give examples of KDF and WDF functions for real occultation event data in the (lower) tropospheric height interval from about 0 to 6 km.

KDF L1	KDF L2
WDF L1	WDF L2

Example results for COSMIC RO events



Example results for Spire RO events



Summary and Conclusions

The Kirkwood Distribution Function (KDF) is evaluated using just one global Fourier Transform. This is much simpler than the evaluation of the Wigner Distribution Function (WDF) and even the spectrogram. Still, it provides a deep insight into the structure of RO signals. Analysis of COSMIC and Spire data with the aid of KDF indicates that it is an effective means of visualizing multipath and reflection patterns.

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